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ON A METHOD OF APPROXIMATE SOLUTION OF THE RIEMANN PROBLEM FOR A ONE-VELOCITY FLOW OF A MULTICOMPONENT MIXTURE

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A method of approximate solution of the Riemann problem on the basis of characteristic relations that can be used in numerical schemes of the Godunov method for integration of equations defining a one-velocity flow of a multicomponent mixture in the adiabatic approximation is presented.

Keywords: *one-velocity flow of a multicomponent mixture, hyperbolic systems of nondivergent form, Riemann problem, numerical simulation.*

Introduction. The Riemann problem for media described by systems of hyperbolic equations can be solved approximately by a number of methods. In the literature, the approximate solvers of Rhie, Harten–Lax–van Leer, Lax–Friedrichs, and others (see [1]) were described. These solvers can be used in the case where an initial system of equations is brought to the divergent form. In the general case, a system of equations describing a one-velocity flow of a multicomponent mixture is not divergent; therefore, there is a need to develop new approaches to solution of the Riemann problem for this medium. One of such approaches is considered in the present work.

Let two homogeneous multicomponent media of infinite mass, consisting of n_L and n_R components each, be located, respectively, at the left and at the right of the plane $x = 0$ at the initial instant of time $t = 0$. The pressures, velocities, densities, and volume fractions of the mixture components in the indicated media are constant and equal to $p_{0(L)}$, $u_{0(L)}$, $\rho_{0(L)}$, $\rho^0_{i0(L)}$, $\alpha_{i0(L)}$ ($i = 1, ..., n_L$) and $p_{0(R)}$, $u_{0(R)}$, $\rho_{0(R)}$, $\rho^0_{j0(R)}$, $\alpha_{j0(R)}$ ($i = 1, ..., n_R$). It is necessary to calculate the flow arising at $t > 0$. The Riemann problem formulated so for a one-velocity flow of the mixture considered in [2] was exactly solved in [3] in the process of cumbersome calculations. In [3], a "rapid" method of approximate solution of the Riemann problem by the finite-difference Courant–Isaacson–Riesz (CIR) method (see [1]) is presented. In the present work, an algorithm of approximate solution of the Riemann problem on the basis of the method of characteristics, which gives much better results as compared to the CIR method, is proposed.

Model of a One-Velocity Flow of a Multicomponent Medium. An n-component mixture with the first m compressed fractions is considered. The system of equations defining the flow of a heterogeneous medium with account for the forces of interfraction interaction has the form (see [2])

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \mathbf{u}) = 0\,, \quad \rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}\right] + \mathrm{grad}\,p = \mathbf{F}\,,$$

$$\frac{\partial}{\partial t}\left[\rho\left(\varepsilon + \frac{1}{2}\,|\mathbf{u}|^2\right)\right] + \mathrm{div}\left[\rho\left(\varepsilon + \frac{1}{2}\,|\mathbf{u}|^2 + \frac{p}{\rho}\right)\mathbf{u}\right] = \mathbf{F}\cdot\mathbf{u}\,,$$

$$\frac{\partial \alpha_i \rho_i^0}{\partial t} + \mathrm{div}\,(\alpha_i \rho_i^0 \mathbf{u}) = \sum_{k=1}^{n(k\neq i)} J_{ik}\,, \qquad (1)$$

$$\rho_i\left(\frac{\partial \varepsilon_i}{\partial t} + (\mathbf{u}\cdot\nabla)\,\varepsilon_i\right) + \frac{\alpha_i p}{\rho_i}\left[\sum_{k=1}^{n(k\neq i)} J_{ik} - \left(\frac{\partial \rho_i}{\partial t} + (\mathbf{u}\cdot\nabla)\,\rho_i\right)\right]$$

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$$= \sum_{k=1}^{n(k \neq i)} (R_{ik} + Q_{ik}) - \left(\varepsilon_i - \frac{1}{2} |\mathbf{u}|^2 \right) \sum_{k=1}^{n(k \neq i)} J_{ik} , \quad i = 1, ..., m - 1 ;$$

$$\frac{\partial \alpha_j}{\partial t} + \text{div}\,(\alpha_j \mathbf{u}) = \frac{1}{\rho_j^0} \sum_{k=1}^{n(k \neq j)} J_{jk} , \quad j = m + 1, ..., n .$$

The behavior of compressed fractions is described by caloric equations of state that have the form $\varepsilon_i = \varepsilon_i(p, \rho_i^0)$ for the ith fraction in the general case; therefore, the expression for the specific internal energy of the mixture

$$\varepsilon = \frac{1}{\rho} \sum_{i=1}^{n} \rho_i \varepsilon_i \tag{2}$$

can be written as

$$\varepsilon = \varepsilon\,(\rho, p, \alpha_1, \rho_1^0, ..., \alpha_{m-1}, \rho_{m-1}^0, \alpha_{m+1}, ..., \alpha_n) . \tag{3}$$

The system of determining equations (1) for one-dimensional plane flows, in which mass forces, phase and chemical transformations, and radiative heat transfer are absent, can be brought, with the use of (3), to the form

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = 0 , \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 ,$$

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \rho c^2 \frac{\partial u}{\partial x} = 0 ,$$

$$\frac{\partial \rho_i^0}{\partial t} + u \frac{\partial \rho_i^0}{\partial x} + \rho_i^0 G_i \frac{\partial u}{\partial x} = 0 , \quad \frac{\partial \alpha_i}{\partial t} + u \frac{\partial \alpha_i}{\partial x} + \alpha_i (1 - G_i) \frac{\partial u}{\partial x} = 0 , \quad i = 1, ..., m - 1 ;$$

$$\frac{\partial \alpha_j}{\partial t} + u \frac{\partial \alpha_j}{\partial x} + \alpha_j \frac{\partial u}{\partial x} = 0 , \quad j = m + 1, ..., n , \tag{4}$$

where $G_i = \frac{1}{\rho_i^0} \left(\frac{\partial \varepsilon_i}{\partial \rho_i^0} \right)^{-1} \left(\frac{p}{\rho_i^0} - \rho c^2 \frac{\partial \varepsilon_i}{\partial p} \right)$. The expression for the velocity of sound in the mixture has the form

$$c = \sqrt{\frac{\frac{p}{\rho} - \rho \frac{\partial \varepsilon}{\partial \rho} - \sum_{i=1}^{m-1} \left[\frac{p}{\rho_i^0} \frac{\partial \varepsilon}{\partial \rho_i^0} \left(\frac{\partial \varepsilon_i}{\partial \rho_i^0} \right)^{-1} + \alpha_i \frac{\partial \varepsilon}{\partial \alpha_i} \left(1 - p \left((\rho_i^0)^2 \frac{\partial \varepsilon_i}{\partial \rho_i^0} \right)^{-1} \right) \right] - \sum_{j=m+1}^{n} \alpha_j \frac{\partial \varepsilon}{\partial \alpha_j}}{\rho \left[\frac{\partial \varepsilon}{\partial p} + \sum_{i=1}^{m-1} \frac{\partial \varepsilon_i}{\partial p} \left(\frac{\partial \varepsilon_i}{\partial \rho_i^0} \right)^{-1} \left(\frac{\alpha_i}{\rho_i^0} \frac{\partial \varepsilon}{\partial \alpha_i} - \frac{\partial \varepsilon}{\partial \rho_i^0} \right) \right]}} . \tag{5}$$

If the behavior of the mixture components is described using the equation of state

$$\varepsilon_i = \frac{p - c_{*i}^2 (\rho_i^0 - \rho_{*i})}{\rho_i^0 (\gamma_i - 1)} = \frac{p B_i + b_i}{\rho_i^0} - d_i , \tag{6}$$

where $B_i = 1/(\gamma_i - 1)$, $d_i = c_{*i}^2 B_i$, $b_i = d_i \rho_{*i}$, expression (2) takes the form

$$\varepsilon = \frac{1}{\rho}\left[pB_m + b_m + \sum_{i=1}^{m-1} \alpha_i\,(pB_{im} + b_{im} - d_{im}\rho_i^0) + \sum_{j=m+1}^{n} \alpha_j \rho_j^0 \varepsilon_j\right] - d_m \,. \tag{7}$$

Here, $B_{im} = B_i - B_m$, $b_{im} = b_i - b_m$, $d_{im} = d_i - d_m$. In this case, the values of G_i and c are determined from the expressions

$$G_i = \frac{\rho c^2 B_i - p}{b_i + pB_i}\,, \quad c = \sqrt{\left\{b_m + p\left[1 + B_m - \sum_{i=1}^{m-1} \frac{\alpha_i\,(b_{im} + pB_{im})}{b_i + pB_i}\right]\right\} \Big/ \left\{\rho\left[B_m + \sum_{i=1}^{m-1} \frac{\alpha_i\,(b_m B_i - b_i B_m)}{b_i + pB_i}\right]\right\}}\ . \tag{8}$$

The characteristic equations of system (4) have only real roots: $u - c$, u, ..., $u + c$ (see [2]). Characteristic relations along the characteristic directions $dx/dt = u \pm c$ can be obtained from the equation

$$\begin{vmatrix}
\xi - u & -\rho & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & -u\frac{d\rho}{dt} - \rho\frac{du}{dt} \\
0 & \xi - u & -\frac{1}{\rho} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & -u\frac{du}{dt} - \frac{1}{\rho}\frac{dp}{dt} \\
0 & -\rho c^2 & \xi - u & 0 & 0 & \dots & 0 & 0 & 0 & \dots & -\rho c^2\frac{du}{dt} - u\frac{dp}{dt} \\
0 & -\rho_1^0 G_1 & 0 & \xi - u & 0 & \dots & 0 & 0 & 0 & \dots & -\rho_1^0 G_1\frac{du}{dt} - u\frac{d\rho_1^0}{dt} \\
0 & \alpha_1(G_1 - 1) & 0 & 0 & \xi - u & \dots & 0 & 0 & 0 & \dots & -\alpha_1(1 - G_1)\frac{du}{dt} - u\frac{d\alpha_1}{dt} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & -\rho_{m-1}^0 G_{m-1} & 0 & 0 & 0 & \dots & \xi - u & 0 & 0 & \dots & -\rho_{m-1}^0 G_{m-1}\frac{du}{dt} - u\frac{d\rho_{m-1}^0}{dt} \\
0 & \alpha_{m-1}(G_{m-1} - 1) & 0 & 0 & 0 & \dots & 0 & \xi - u & 0 & \dots & -\alpha_{m-1}(1 - G_{m-1})\frac{du}{dt} - u\frac{d\alpha_{m-1}}{dt} \\
0 & -\alpha_{m+1} & 0 & 0 & 0 & \dots & 0 & 0 & \xi - u & \dots & -\alpha_{m+1}\frac{du}{dt} - u\frac{d\alpha_{m+1}}{dt} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & -\alpha_n & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & -\alpha_n\frac{du}{dt} - u\frac{d\alpha_n}{dt}
\end{vmatrix} = 0,$$

where $\xi = dx/dt$. Calculation of the determinant gives the following expressions true along the characteristic directions $dx/dt = u \pm c$:

$$dp \pm \rho c du = 0\,. \tag{9}$$

Along the trajectory characteristic $dx/dt = u$, the equations

$$dp - c^2 d\rho = 0\,, \quad d\alpha_i - \frac{\alpha_i\,(1 - G_i)}{\rho}\, d\rho = 0\,, \quad d\rho_i - \frac{\rho_i}{\rho}\, d\rho = 0\,, \quad i = 1, ..., m - 1\,; \tag{10}$$

$$d\rho - \frac{\rho}{\alpha_j}\, d\alpha_j = 0\,, \quad j = m + 1, ..., n\,, \tag{11}$$

following from system (4), are fulfilled. Integrating the latter relations of (10) and expression (11), we obtain

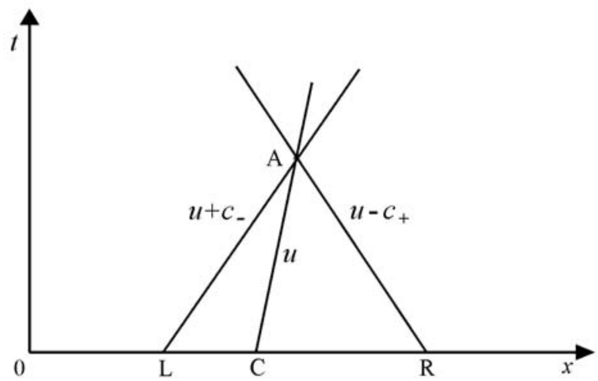

Fig. 1. Computational scheme.

$$\rho_i^0 = \rho_{i0}^0 \frac{\alpha_{i0}}{\alpha_i} \frac{\rho}{\rho_0}, \quad \alpha_j = \alpha_{j0} \frac{\rho}{\rho_0}. \tag{12}$$

Algorithm of Approximate Solution of the Riemann Problem. We now consider a method of approximate solution of the Riemann problem by the method of characteristics. The values of the parameters at the contact boundary will be determined with the use of an iteration procedure. For this purpose, the trajectory characteristic $dx/dt = u$ will be merged from the point C positioned at the contact boundary (see Fig. 1). From the point A lying on this straight line, the "right" characteristic $dx/dt = u + c_-$ and the "left" characteristic $dx/dt = u - c_+$ will be merged and continued to the intersection with the coordinate line $t = 0$ at the points L and R, at which the values of variables are determined from the initial conditions. Rewriting the differential relations (9)–(12), true along the indicated characteristic directions AL, AR, and AC, in the finite-difference form, we obtain

$$p^{(s+1)} - p_{0(\mathrm{L})} + a_1 (u^{(s+1)} - u_{0(\mathrm{L})}) = 0, \quad p^{(s+1)} - p_{0(\mathrm{L})} - a_2 (\rho_-^{(s+1)} - \rho_{0(\mathrm{L})}) = 0,$$

$$\alpha_{i-}^{(s+1)} - \alpha_{i0(\mathrm{L})} - a_3 (\rho^{(s+1)} - \rho_{0(\mathrm{L})}) = 0, \quad \rho_{i-}^{0(s+1)} = \rho_{i0(\mathrm{L})}^0 \frac{\alpha_{i0(\mathrm{L})}\rho_-^{(s+1)}}{\alpha_{i-}^{(s+1)}\rho_{0(\mathrm{L})}}, \quad i = 1, ..., m_{\mathrm{L}} - 1;$$

$$\alpha_j^{(s+1)} = \alpha_{j0(\mathrm{L})} \frac{\rho_-^{(s+1)}}{\rho_{0(\mathrm{L})}}, \quad j = m_{\mathrm{L}} + 1, ..., n_{\mathrm{L}}; \tag{13}$$

$$p^{(s+1)} - p_{0(\mathrm{R})} - a_4 (u^{(s+1)} - u_{0(\mathrm{R})}) = 0, \quad p^{(s+1)} - p_{0(\mathrm{R})} - a_5 (\rho_+^{(s+1)} - \rho_{0(\mathrm{R})}) = 0,$$

$$\alpha_{i+}^{(s+1)} - \alpha_{i0(\mathrm{R})} - a_6 (\rho_+^{(s+1)} - \rho_{0(\mathrm{R})}) = 0, \quad \rho_{i+}^{0(s+1)} = \rho_{i0(\mathrm{R})}^0 \frac{\alpha_{i0(\mathrm{R})}\rho_+^{(s+1)}}{\alpha_{i+}^{(s+1)}\rho_{0(\mathrm{R})}}, \quad i = 1, ..., m_{\mathrm{R}} - 1;$$

$$\alpha_j^{(s+1)} = \alpha_{j0(\mathrm{R})} \frac{\rho_+^{(s+1)}}{\rho_{0(\mathrm{R})}}, \quad j = m_{\mathrm{R}} + 1, ..., n_{\mathrm{R}}.$$

Here,

$$a_1 = \frac{1}{2}((\rho c)_{0(\mathrm{L})} + (\rho c)_-^{(s)}), \quad a_2 = \frac{1}{2}(c_{0(\mathrm{L})}^2 + (c_-^2)^{(s)}), \quad a_3 = \frac{1}{2}\left(\frac{\alpha_{i0(\mathrm{L})}(1 - G_{i0(\mathrm{L})})}{\rho_{0(\mathrm{L})}} + \frac{\alpha_{i-}^{(s)}(1 - G_{i-}^{(s)})}{\rho_-^{(s)}}\right),$$

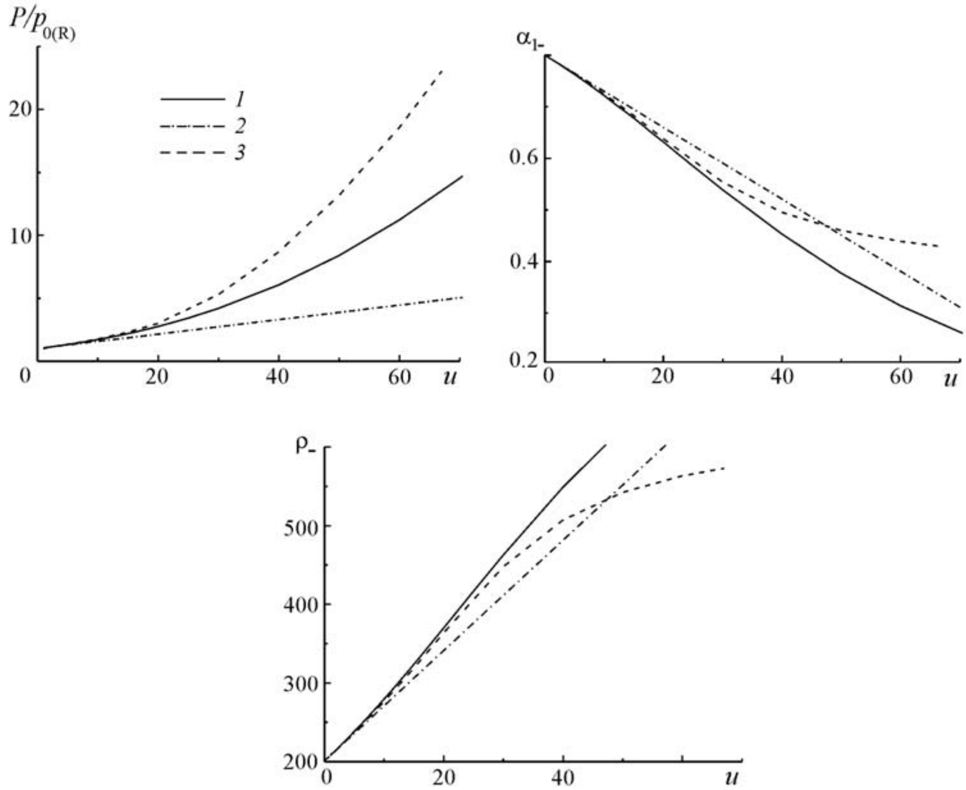

Fig. 2. Dependence of the parameters $P/p_{0(R)}$, α_{1-}, and ρ_- on u for the regime of flow with two shock waves: 1) exact values; 2, 3) approximate values obtained with the use of the CIR relations and relation (13) respectively.

$$a_4 = \frac{1}{2}\left((\rho c)_{0(R)} + (\rho c)_+^{(s)}\right), \quad a_5 = \frac{1}{2}\left(c_{0(R)}^2 + (c_+^2)^{(s)}\right), \quad a_6 = \frac{1}{2}\left(\frac{\alpha_{i0(R)}(1 - G_{i0(R)})}{\rho_{0(R)}} + \frac{\alpha_{i+}^{(s)}(1 - G_{i+}^{(s)})}{\rho_+^{(s)}}\right).$$

The symbols "+" and "-" denote the parameters on the "right" and "left" sides of the contact boundary respectively. The iteration process used for calculations by formulas (13) converges fairly rapidly, which allows one to determine the values of the indicated parameters. Note that the pressure and velocity of the flow at the contact boundary are continuous, i.e., $p_- = p_+ = P$ and $u_- = u_+ = U$. The other characteristics, such as the velocity of movement of shocks, can be calculated with the use of the Rankine–Hugoniot relations (see [2]).

Let us consider the application of the above-described algorithm for approximate solution of the Riemann problem for a gas-liquid mixture with two compressed fractions, in which the breaking of an arbitrary shock occurs.

Figure 2 shows dependences of the relative pressure $P/p_{0(L)}$, the volume fraction α_{1-}, and the density ρ_- at the contact boundary between two flows of identical gas-liquid media on the rate of their interaction ($\gamma_{1(L)} = \gamma_{1(R)} = 1.4$, $c_{*1(L)} = c_{*1(R)} = 0$, $\rho^0_{20(L)} = \rho^0_{20(R)} = 1.19$ kg/m^3, $\gamma_{2(L)} = \gamma_{2(R)} = 5.59$, $c_{*2(L)} = c_{*2(R)} = 1500$ m/sec, $\rho_{*2(L)} = \rho_{*2(R)} = 1000$ kg/m^3, $u_{0(R)} = -u_{0(L)}$, $u_{0(L)} > 0$) at $p_{0(L)} = p_{0(R)} = 10^5$ Pa, $\alpha_{10(L)} = \alpha_{10(R)} = 0.8$, $\rho^0_{10(L)} = \rho^0_{10(R)} = 1.19$ kg/m^3, $\rho^0_{20(L)} = \rho^0_{20(R)} = 1000$ kg/m^3. As a result of the collision of these flows, a symmetric flow with two shock waves is formed. Exact values of the parameters at the boundary of the breaking of the shock were calculated with the use of the algorithm presented in [3] (curves 1); approximate solutions were obtained by the CIR method (curves 2) with the use of formulas (13) (curves 3).

As an example of a flow including a shock wave and a rarefaction wave, we considered the Riemann problem with the following initial data: $p_{0(L)} = p_{0(R)} = 0.1$ MPa, $u_{0(L)} = u_{0(R)} = 0$, $\alpha_{10(L)} = \alpha_{10(R)} = 0.9$. The mixture components are identical to those considered in the previous problem. Figure 3 presents the dependences $P/p_{0(R)}$ and U

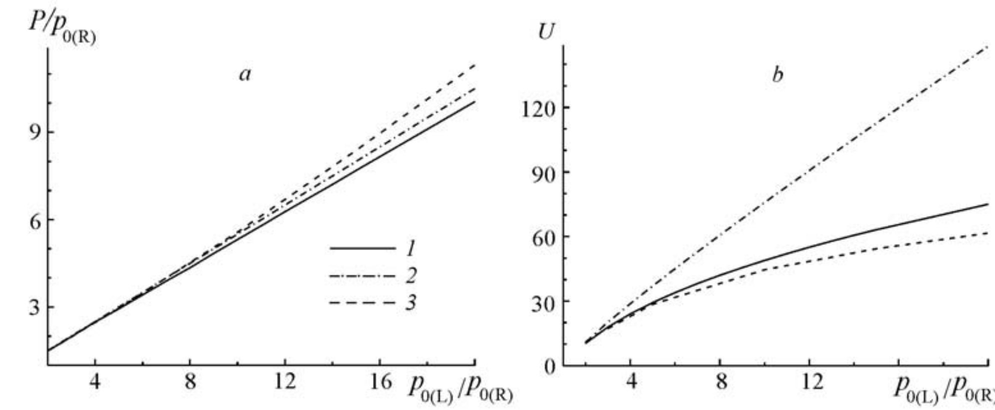

Fig. 3. Dependence of the parameters $P/p_{0(R)}$ and U on the pressure drop $p_{0(L)} = p_{0(R)}$ for the regime with a shock-wave and a rarefaction wave. Designations 1–3 are identical to those in Fig. 2.

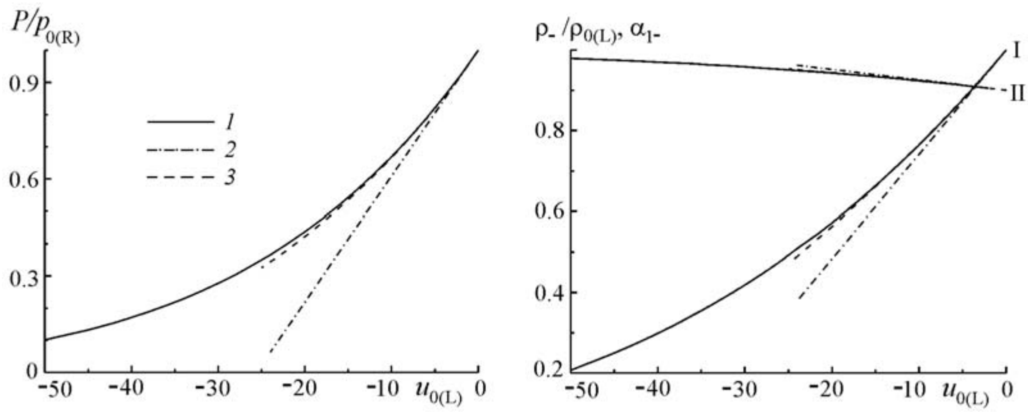

Fig. 4. Dependences of the parameters $P/p_{0(R)}$, $\rho_-/\rho_{0(L)}$ (I), and α_{1-} (II) on $u_{0(L)}$ for the regime of flow with two rarefaction waves. Designations 1–3 are identical to those in Fig. 2.

on the initial pressure drop $p_{0(L)}/p_{0(R)}$, calculated by the formulas providing an exact and an approximate solution of the Riemann problem.

Figure 4 shows the dependences $P/p_{0(R)}$, $\rho/\rho_{0(L)}$, and α_{1-} on the rate of interaction of flows for the problem on symmetric scattering ($u_{0(L)} < 0$, $u_{0(R)} = -u_{0(L)}$) of gas-liquid media under the following initial conditions: $p_{0(L)} = p_{0(R)} = 10^5$ Pa, $\alpha_{10(L)} = \alpha_{10(R)} = 0.9$, $\rho^0_{10(L)} = \rho^0_{10(R)} = 1.19$ kg/m^3, $\rho^0_{20(L)} = \rho^0_{20(R)} = 1000$ kg/m^3. In this case, as a result of the breaking of the shock, a flow with two rarefaction waves arises.

It is seen from Figs. 2–4 that, if the initial parameters of the flows on the different sides of the contact boundary do not differ strongly from each other, the results of calculation of the breaking of the shock, obtained by the exact expressions and approximate expressions (13), are close. Thus, the application of the approximate solver, based on the characteristic relations used in numerical solutions by the method of S. K. Godunov, to the solution of the Riemann problem will be warranted if calculations are carried out on fairly fine finite-difference grids. If coarse grids are used, in calculations carried out by the Godunov method, formulas providing an exact solution of the Riemann problem from [3] should be used.

Conclusions. A "rapid" method of approximate solution of the Riemann problem for the flow of a one-velocity multicomponent mixture in the adiabatic approximation on the basis of characteristic relations has been developed. This method can be used in the numerical Godunov schemes for integration of equations defining this mixture. It was shown that the method proposed allows one to obtain a more exact solution of the Riemann problem as compared to that obtained by the CIR method.

NOTATION

c, adiabatic velocity of sound in a mixture; c_{*i}, constant of the equation of state; $\mathbf{F}$, density of the mass force; J_{ij}, intensity of transformation of the mass of a unit volume of the mixture from the ith fraction into the jth fraction; p, pressure of the gas-liquid medium; P, pressure at a contact discontinuity; Q_{ij}, heat released in a unit time by a unit volume of the mixture due to the transformation of the ith fraction into the jth fraction; R_{ij}, heat released in a unit time by a unit volume of the mixture that is transferred from the jth fraction to the ith one by radiation; t, time; u, velocity of the gas-liquid medium; $\mathbf{u}$, velocity vector; U, velocity at the contact discontinuity; x, spatial variable; α_i, volume fraction of the ith fraction; γ_i, constant of the equation of state; ε_i, specific internal energy; ρ, density of the mixture; ρ_i^0, real density of the ith fraction; $\rho_i = \alpha_i \rho_i^0$, reduced density of the ith component; ρ_{*i}, constant of equation of state. Subscripts: 0, in the undisturbed medium; (L), (R), for the mixture parameters at the left and at the right of the contact discontinuity; s, number of an iteration; i, j, numbers of fractions; n, total number of mixture components; m, number of compressed fractions.

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